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Quantum Trajectories, State Diffusion, and Time-Asymmetric Eventum Mechanics

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We show that the quantum stochastic Langevin model for continuous in time measurements provides an exact formulation of the von Neumann uncertainty error-disturbance principle. Moreover, as it was shown in the 1980s, this Markov model induces all stochastic linear and nonlinear equations of the phenomenological informational dynamics such as quantum state diffusion and spontaneous localization by a simple quantum filtering method. Here we prove that the quantum Langevin equation is equivalent to a Dirac-type boundary-value problem for the second quantized input "offer waves from future" in one extra dimension, and to a reduction of the algebra of the consistent histories of past events to an Abelian subalgebra for the "trajectories of the output particles." This result supports the wave–particle duality in the form of the thesis of Eventum Mechanics that everything in the future is constituted by quantized waves, everything in the past by trajectories of the recorded particles. We demonstrate how this time arrow can be derived from the principle of quantum causality for nondemolition continuous in time measurements.

KEY WORDS: quantum trajectories; state diffusion; eventum mechanics.

1. INTRODUCTION

> Quantum mechanics itself, whatever its interpretation, does not account for the transition from "possible to the actual"
>
> —Heisenberg

Schrödinger believed that all problems of interpretation of quantum mechanics including the above problem for time arrow should be formulated in continuous time in the form of differential equations. He thought that the quantum jump problem would have been resolved if quantum mechanics had been made consistent with relativity theory of events and the time had been treated appropriately as a future–past boundary value problem of a microscopic information

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dynamics. However Einstein and Heisenberg did not believe this, each for his own reasons.

Although Schrödinger did not succeed in finding the "true Schrödinger equation" so he could formulate the boundary value problem for such "eventum mechanics," the analysis of the phenomenological stochastic models for quantum diffusions and spontaneous jumps proves that Schrödinger was right. We shall see that there exists indeed a boundary value problem for the "true Schrödinger equation" which corresponds to quantum jumps and diffusive trajectories which is as continuous as Schrödinger could have wished, but it is not the usual Schrödinger, but an ultrarelativistic (massless) Dirac-type boundary value problem in second quantization. However Heisenberg was also right, as to take into account for these transitions by filtering the actual past events simply as it is done in classical statistics, the corresponding Dirac-type boundary value problem must be supplemented by future-past supers-election rule for the total algebra as it follows from the nondemolition causality principle (Belavkin, 1994). This principle demands the arrow of time, and it cannot be formulated in the orthodox quantum mechanics as it involves infinitely many degrees of freedom, and is yet unknown even in the quantum field theory.

Here we shall deal with quantum white noise models which allow us to formulate the most general stochastic decoherence equation which was derived in Belavkin (1995b) from the unitary quantum Langevin equation. We shall start with a simple quantum noise model and show that it allows us to prove the "true Heisenberg principle" in the form of an uncertainty relation for measurement errors and dynamical perturbations. The discovery of quantum thermal noise and its white-noise approximations lead to a profound revolution not only in modern physics but also in contemporary mathematics comparable with the discovery of differential calculus by Newton (for a feature exposition of this, accessible for physicists, see Gardiner (1991), the complete theory, which was mainly developed in the 1980s (Belavkin, 1980, 1988b; Gardiner, 1985; Hudson and Parthasarthy, 1984), is sketched in the Appendix). Then we formulate the corresponding boundary value problem of the Eventum Mechanics—the extended quantum mechanics with a superselection causality rule in which there is a place for microscopic events and trajectories. The dynamics of this event-enhanced quantum mechanics is described by a one-parametric group of unitary propagators on an extended Hilbert space, as in the conventional quantum mechanics, however it is *essentially irreversible*, as the induced Heisenberg dynamics forms only a semigroup of *invertible endomorphisms* (but not of automorphisms!) in the positive arrow of time chosen by the causality.

During the 1990s many "primary" quantum theories appeared in the theoretical and applied physics literature, in particular, the quantum state diffusion theory (Gisin and Percival, 1992, 1993), where a particular type of the nonlinear quantum filtering stochastic equation has been used without even a reference to the continuous measurements. The recent phenomenological models for quantum trajectories in quantum optics (Carmichael, 1993, 1994; Goetsch and Graham, 1993, 1994; Wiseman and Milburn, 1993, 1994) are also based on the stochastic solutions to quantum jump equations, although the underlying boundary value problems of eventum mechanics and the corresponding quantum stochastic filtering equations of mathematical physics remain largely unknown in the general physics. An exception occurred only in Goetsch and Graham (1994, 1995), where our quantum stochastic filtering theory which had been developed for these purposes in the 1980s, was well understood at both a macroscopic and microscopic level. We complete this paper by formulating and discussing the basic principles of the eventum mechanics as microscopic time-asymmetric information dynamics, which may include also classical mechanics, and which is consistent with the quantum decoherence and quantum measurement.

2. THE TURE HEISENBERG PRINCIPLE

The first, time-continuous solution of the quantum measurement problem (Belavkin, 1980) was motivated by analogy with the classical stochastic filtering problem which obtains the prediction of the future for an unobservable dynamical process $x(t)$ by time-continuous measuring of another, observable process $y(t)$. Such problems were first considered by Wiener and Kolmogorov, who found the solutions in the form of a causal spectral filter for a linear estimate $\hat{x}(t)$ of $x(t)$, which is optimal only in the stationary Gaussian case. The complete solution of this problem was obtained by Stratonovich (1966) in 1958, who derived a stochastic filtering equation giving the posterior expectations $\hat{x}(t)$ of $x(t)$ in the arbitrary Markovian pair (x, y). This was really a breakthrough in the statistics of stochastic processes which soon found many applications, in particular, for solving the problems of stochastic control under incomplete information (it is possible that this was one of the reasons why the Russians were so successful in launching the rockets to the moon and other planets of the solar system in 1960s).

If $X(t)$ is an unobservable Heisenberg process, or vector of such processes $X_k(t)$, $k = 1, \ldots, d$, which might even have no prior trajectories as the Heisenberg coordinate processes of a quantum particle say, and $Y(t)$ is an actual observable quantum processes, i.e., a sort of Bell's beable describing the vector trajectory y (t) of the particle in a cloud chamber say, why do we not find the posterior trajectories by deriving and solving a filtering equation for the posterior expectations $\hat{x}(t)$ of $X(t)$ or any other function of $X(t)$, defining the posterior trajectories $x(t, y_0^t)$ in the same way as we do it in the classical case? If we had a dynamical model in which such beables existed as a nondemolition process, we could solve this problem simply by conditioning as the statistical inference problem, predicting the future knowing a history, i.e., a particular trajectory $y(r)$ up to the time t. This problem was first considered and solved by finding a nontrivial quantum stochastic model for the Markovian Gaussian pair (X, Y). It corresponds to a quantum open linear system with linear output channel, in particular for a quantum oscillator matched to a quantum transmission line (Belavkin, 1980, 1985). By studying this example, the nondemolition condition,

$$[X_\kappa(s), Y(r)] = 0, \quad [Y(s), Y(r)] = 0 \qquad \forall r \leq s,$$

was first found, and this allowed the solution in the form of the causal equation for $x(t, y_0^{t]}) = \langle X(t)\rangle_{y_0^{t]}}$.

Let us describe this exact dynamical model of the causal nondemolition measurement first in terms of quantum white noise for a quantum nonrelativistic particle of mass m which is conservative, if not observed, in a potential field ϕ. But we shall assume that this particle is under a time-continuous indirect observation which is realized by measuring of its Heisenberg position operators $Q^\kappa(t)$ with additive random errors $e^\kappa(t)$:

$$Y^\kappa(t) = Q^\kappa(t) + e^\kappa(t), \qquad \kappa = 1, \ldots, d.$$

We take the simplest statistical model for the error process $e(t)$, the white-noise model (the worst, completely chaotic error), assusming that it is a classical Gaussian white noise given by the first momenta

$$\langle e^\kappa(t)\rangle = 0, \qquad \langle e^\kappa(s)e^l(r)\rangle = \sigma_e^2\delta(s-r)\delta_l^\kappa.$$

The components of measurement vector-process $Y(t)$ should be commutative, satisfying the causal nondemolition condition with respect to the noncommutative process $Q(t)$ (and any other Heisenberg operator-process of the particle), this can be achieved by perturbing the particle Newton–Ehrenfest equation:

$$m\,\frac{d^2}{dt^2}\,Q(t) + \nabla\phi(Q(t)) = f(t).$$

Here $f(t)$ is vector process of Langevin forces f_κ perturbing the dynamics due to the measurement, which are also assumed to be independent classical white noises

$$\langle f_\kappa(t)\rangle = 0, \qquad \langle f_\kappa(s) f_l(r)\rangle = \sigma_f^2\delta(s-r)\delta_l^\kappa.$$

In classical measurement and filtering theory the white noises $e(t)$, $f(t)$ are usually considered independent, and the intensities σ_e^2 and σ_f^2 can be arbitrary, even zeros, corresponding to the ideal case of the direct unperturbing observation of the particle trajectory $Q(t)$. However in quantum theory corresponding to the standard commutation relations,

$$Q(0) = \mathrm{Q}, \quad \frac{\mathrm{d}}{\mathrm{d}t}Q(0) = \frac{1}{m}\mathrm{P}, \quad [\mathrm{Q}^\kappa, \mathrm{P}_l] = i\hbar\delta_l^\kappa\mathrm{I},$$

the particle trajectories do not exist such that the measurement error $e(t)$ and parturbation force $f(t)$ should satisfy a sort of uncertainty relation. This "true

Heisenberg principle" had never been mathematically proved before the discovery (Belavkin, 1980) of quantum causality in the form of nondemolition condition of commutativity of $Q(s)$, as well as any other process, the momentum $P(t) = m\dot{Q}(t)$ say, with all $Y(r)$ for $r \leq s$. As we showed first in the linear case (Belavkin, 1980, 1985), and later even in the most general case (Belavkin, 1992b), these conditions are fulfilled if and only if $e(t)$ and $f(t)$ satisfy the cononical commutation relations

$$[e^\kappa(r), e^l(s)] = 0, [e^\kappa(r),\ f_l(s)] = \frac{\hbar}{i}\delta(r-s)\delta^\kappa_l, [\,f_\kappa(r),\ f_l(s)] = 0.$$

From this it follows that the pair $(e,\ f)$ satisfies the uncertainty relation $\sigma_e\sigma_f \geq \hbar/2$. This inequality constitutes the precise formulation of the true Heisenberg principle for the square roots σ_e and σ_f of the intensities of error e and perturbation f: they are inversely proportional with the same coefficient of proportionality, $\hbar/2$, as for the pair (Q, P). Note that the canonical pair $(e,\ f)$ called quantum white noise cannot be considered classically, despite the fact that each process e and f separately can. This is why we need a quantum-field representation for the pair $(e,\ f)$, and the corresponding quantum stochastic calculus. Thus, a generalized matrix mechanics for the treatment of quantum open systems under continuous nondemolition observation and the true Heisenberg principle was discovered 20 years ago only after the invention of quantum white noise in (Belavkin, 1980). The nondemolition commutativity of $Y(t)$ with respect to the Heisenberg operators of the open quantum system was later rediscovered for the output of quantum stochastic fields in Gardiner and Collett (1985).

Let us outline the exact quantum stochastic model (Belavkin, 1988a, 1992b) for a quantum particle of mass m in a potential ϕ under indirect observation of the positions Q^κ by measuring Y_κ. We define the output process as a quantum stochastic Heisenberg transformation $Y^t_\kappa = W(t)^\dagger(\mathrm{I} \oplus \hat{y}^t_\kappa)W(t)$ for a time-continuous quantum stochastic unitary evolution $W(t)$. It has been shown in (Belavkin, 1988a, 1992b) that $W(t)$ is the resolving family for an appropriate *quantum stochastic Schrödinger equation* (see Eq. (3) below). It induces the following quantum stochastic Heisenberg output equation:

$$dY^t_\kappa = 2\lambda Q^\kappa(t)\,\mathrm{d}t + \mathrm{d}\hat{w}^t_\kappa \equiv X_\kappa(t)\,\mathrm{d}t + \mathrm{d}\hat{w}^t_\kappa, \tag{1}$$

where λ is a coupling constant, or a diagonal matrix $\lambda = [\lambda_\kappa\delta^l_\kappa]$ defining different accuracies of an indirect measurement at time of Q^κ. Here $X(t) = W(t)^\dagger(\mathrm{X} \otimes I_0)W(t)$ are the system Heisenberg operators for $\mathrm{X}_\kappa = 2(\lambda Q)^\kappa$, I_0, is the identity operator in the Fock space $\mathcal{F}_0$, and $\hat{w}^t_\kappa \equiv y^t_\kappa$, $\kappa = 1, \ldots, d$ are the standard independent Wiener processes w^t_κ represented as the operators $\hat{w}^t_\kappa = A^\kappa_-(t) + A^+_\kappa(t)$ on the Fock vacuum vector $\delta_\emptyset \in \mathcal{F}_0$ such that $w^t_\kappa \simeq \hat{w}^t_\kappa\delta_o$ (see the notations and more about the quantum stochastic calculus in Fock space in the Appendix). This model coincides with the signal plus noise model given above if

$$\hat{e}^{\kappa}(t) = \frac{1}{2}(a_{\kappa}^{+} + a_{-}^{\kappa})(t) = \frac{1}{2\lambda_{\kappa}} \frac{\mathrm{d}w_{\kappa}^{t}}{\mathrm{d}t},$$

where $a_{\kappa}^{+}(t), a_{-}^{\kappa}(t)$ are the canonical bosonic creation and annihilation field operators,

$$[a_{\kappa}^{+}(s), a_{l}^{+}(t)] = 0, [a_{-}^{\kappa}(s), a_{l}^{+}(t)] = \delta_{l}^{\kappa}\delta(t-s), [a_{-}^{\kappa}(s), a_{-}^{l}(t)] = 0,$$

defined as the generalized derivatives of the standard quantum Brownian motions $A_{\kappa}^{+}(t)$ and $A_{-}^{\kappa}(t)$ in Fock space $\mathcal{F}_0$. It was proved in (Belavkin, 1988a, 1992b) that Y_{κ}^{t} is a commutative nondemolition process with respect to the system Heisenberg coordinate and momentum $P(t) = W(t)^{\dagger}(\mathrm{P} \otimes I)W(t)$ processes if they are perturbed by independent Langevin forces $f_{\kappa}(t)$ of intensity $\tau_{\kappa}^{2} = (\lambda_{\kappa}\hbar)^{2}$, the generalized derivatives of $f_{\kappa}^{t} \simeq \hat{f}_{\kappa}^{t}\delta_{\emptyset}$ times λ_{κ}, where $\hat{f}_{\kappa}^{t} = i\hbar(A_{-}^{\kappa}A_{\kappa}^{+})(t)$:

$$\mathrm{d}P_{\kappa}(t) + \phi_{\kappa}'(Q(t))\,\mathrm{d}t = \lambda_{\kappa}\,\mathrm{d}\hat{f}_{\kappa}^{t}, \quad P_{\kappa}(t) = m\frac{\mathrm{d}}{\mathrm{d}t}Q^{\kappa}(t). \tag{2}$$

Note that the quantum error operators $\hat{w}_{\kappa}^{t}$ commute, but they do not commute with the perturbing quantum force operators $\hat{f}_{\kappa}^{t}$ in Fock space due to the multiplication table

$$(\mathrm{d}\hat{w}_{\kappa})^{2} = I\,\mathrm{d}t, \quad \mathrm{d}\hat{f}_{\kappa}\,\mathrm{d}\hat{w}_{l} = i\hbar I\delta_{l}^{\kappa}\,\mathrm{d}t,$$
$$\mathrm{d}\hat{w}_{\kappa}\,\mathrm{d}\hat{f}_{l} = -i\hbar I\delta_{l}^{\kappa}\,\mathrm{d}t, \quad (\mathrm{d}\hat{f}_{\kappa})^{2} = \hbar^{2}I\,\mathrm{d}t.$$

This corresponds to the cononical commutation relations for the renormalized derivatives $\hat{w}_{\kappa}(t)$ and $\hat{f}_{l}(t)$, so that the true Heisenberg principle is fulfilled at the boundary $\sigma_{\kappa}\tau_{\kappa} = \hbar/2$. Thus our quantum stochastic model of nondemolition observation is the minimal perturbation model for the given accuracy λ of the continual indirect measurement of the position operators $Q(t)$ (the perturbation vanishes when $\lambda = 0$).

3. QUANTUM STATE DIFFUSION AS INFORMATION DYNAMICS

Let us introduce the quantum stochastic wave equation for the unitary transformation $\Psi_0(t) = W(t)\Psi_0$ inducing Heisenberg dynamics which is described by the quantum Langevin Eq. (2) with white-noise perturbation. This equation is well understood in terms of the generalized derivatives

$$\hat{f}_{\kappa}(t) = \lambda_{\kappa}\frac{\hbar}{i}(a_{\kappa}^{+} - a_{-}^{\kappa})(t) = \lambda_{\kappa}\frac{\mathrm{d}\hat{f}_{\kappa}^{t}}{\mathrm{d}t}$$

of the standard quantum Brownian motions $A_\kappa^+(t)$ and $A_-^\kappa(t)$ defined by the commutation relations

$$[A_\kappa^+(s), A_l^+(t)] = 0, [A_-^\kappa(s), A_l^+(t)] = (t \wedge s)\delta_l^\kappa, [A_-^\kappa(s), A_-^l(t)] = 0$$

in Fock space $\mathcal{F}_0(t \wedge s = \min\{s, t\})$. The corresponding quantum stochastic differential equation for the probability amplitude in $\mathfrak{h} \otimes \mathcal{F}_0$ is a particular case

$$\mathrm{L}_\kappa^{-\dagger} - \mathrm{L}_+^\kappa, \qquad \mathrm{L}_+^\kappa = (\lambda \mathrm{Q})^\kappa \equiv \mathrm{L}^\kappa$$

of the general quantum diffusion wave equation,

$$\mathrm{d}\Psi_0(t) + (\mathrm{K} \otimes I)\Psi_0(t)\,\mathrm{d}t = (\mathrm{L}_+^\kappa \otimes \mathrm{d}A_\kappa^+ + \mathrm{L}_\kappa^- \otimes \mathrm{d}A_-^\kappa)(t)\Psi_0(t), \tag{3}$$

which describes the unitary evolution in $\mathfrak{h} \otimes \mathcal{F}_0$ if $\mathrm{K} = \frac{i}{\hbar}\mathrm{H} - \frac{1}{2}\mathrm{L}_\kappa^-\mathrm{L}_+^\kappa$, where $\mathrm{H}^\dagger = \mathrm{H}$ is the evolution Hamiltonian for the system in $\mathfrak{h}$. Using the quantum Itô formula (see the Appendix), it was proven in (Belavkin, 1988a, 1992b) that it is equivalent to the Langevin equation

$$\begin{aligned}\mathrm{d}X(t) &= (f(XL + L^\dagger X) + L^\dagger XL - K^\dagger X - XK)(t)\,\mathrm{d}t \\ &\quad + (fX + L^\dagger X - XL)(t)\,\mathrm{d}A_- + (fX + XL - L^\dagger X)(t)\,\mathrm{d}A^+ \end{aligned} \tag{4}$$

for any quantum stochastic Heisenberg process

$$X(t, f) = W(t)^\dagger \left(\mathrm{X} \otimes \exp\left[\int_0^t \left(f^\kappa(r)\,\mathrm{d}\hat{w}_\kappa^r - \frac{1}{2} f(r)^2\,\mathrm{d}r\right)\right]\right) W(t),$$

where $f^\kappa(t)$ are a test function for the output process w_κ^t and

$$K(t) = W(t)^\dagger(\mathrm{K} \otimes \mathrm{I})W(t), \quad L^\kappa(t) = W(t)^\dagger(L^\kappa \otimes \mathrm{I})W(t).$$

The Langevin Eq. (2) for the system coordinate $X(t) = W(t)^\dagger \cdot (\mathrm{Q} \otimes \mathrm{I})W(t)$ and for the output processes $Y(t, f)$ corresponding to $\mathrm{X} = \mathrm{I}$ follows straightforwardly in the case $\mathrm{L} = \lambda\mathrm{Q}$, $\mathrm{H} = \mathrm{P}^2/2m + \phi(\mathrm{Q})$.

In the next section we shall show that this unitary evolution is the interaction picture for a unitary group evolution U^t corresponding to a Dirac-type boundary value problem for a generalized Schrödinger equation in an extended product Hilbert space $\mathfrak{h} \otimes \mathcal{G}$. Here we prove that the quantum stochastic evolution (3) in $\mathfrak{h} \otimes \mathcal{F}_0$ coincides with the quantum state diffusion in $\mathfrak{h}$ if it is considered only for the initial product states $\psi \otimes \delta_\emptyset$ with $\delta_\emptyset$ being the Fock vacuum state vector in $\mathcal{F}_0$.

Quantum state diffusion is a nonlinear, nonunitary, irreversible stochastic form of quantum mechanics with trajectories put forward by Gisin and Percival (1992, 1993) in the early 1990s as a new, *primary* quantum theory which includes the diffusive reduction process into the wave equation for pure quantum states. It has been criticized, quite rightly, as an incomplete theory which does not satisfy the linear superposition principle for the waves, and for not explaining the origin of irreversible dissipativity which is built into the equation "by hand." In fact the "primary" equation had been derived even earlier as the posterior state diffusion equation for pure states $\psi_\omega = \psi(\omega)/\|\psi(\omega)\|$ from the linear unitary quantum diffusion Eq. (3) by the following method as a particular type of the general quantum filtering equation in the literature (Belavkin, 1988a, 1989; Belavkin and Staszewski, 1992). Here we shall show only how to derive the corresponding stochastic linear decoherence equation for $\psi(t, w) = V(t, w) = V(t, w)\psi$ when all the independent increment processes y_κ^t are of the diffusive type $y_\kappa^t = w_\kappa^t$:

$$\mathrm{d}\psi(t, \omega) + \mathrm{K}\psi(t, \omega)\,\mathrm{d}t = \mathrm{L}^\kappa \psi(t, \omega)\,\mathrm{d}w_\kappa^t, \quad \psi(0) = \psi. \tag{5}$$

Note that the resolving stochastic propagator $V(t, \omega)$ for this equation defines the isometries

$$V(t)^\dagger V(t) = \int V(t, \omega)^\dagger V(t, \omega)\,\mathrm{d}\mu = 1$$

of the system Hilbert space $\mathfrak{h}$ into the Wiener–Hilbert space L^2_μ of square integrable functionals of the diffusive trajectories $\omega = \{\omega(t)\}$ with respect to the standard Gaussian measure $\mu = \mathsf{P}_w$ if $\mathrm{K} + \mathrm{K}^\dagger = \mathrm{L}^\dagger\mathrm{L}$.

Let us represent these Wiener processes in the equation by operators $\hat{w}_\kappa^t = A_\kappa^+ + A_-^\kappa$ on the Fock space vacuum $\delta_\emptyset$, using the unitary equivalence $w_\kappa^t \simeq \hat{w}_\kappa^t \delta_\emptyset$ in the notation explained in the Appendix. Then the corresponding operator equation,

$$\mathrm{d}\hat{\psi}(t) + \mathrm{K}\hat{\psi}(t)\,\mathrm{d}t = (\mathrm{L}^\kappa\,\mathrm{d}A_\kappa^+ + L_\kappa^\dagger\,\mathrm{d}A_-^\kappa)\hat{\psi}(t), \quad \hat{\psi}(0) = \psi \otimes \delta_\phi,\ \psi \in \mathfrak{h},$$

with $\mathrm{L}_\kappa^\dagger = \lambda \mathrm{Q}^\kappa = \mathrm{L}^\kappa$ coincides with the quantum diffusion Schrödinger Eq. (3), where $\mathrm{L}_+^\kappa = \mathrm{L}^\kappa$, $\mathrm{L}_\kappa^- = -\mathrm{L}_\kappa$ on the same initial product-states $\Psi_0(0) = \psi \otimes \delta_\emptyset$. Indeed, as it was noted in Belavkin (1992b), due to the adaptedness,

$$\hat{\psi}(t) = \hat{\psi}^t \otimes \delta_\emptyset, \qquad \Psi_0(t) = \Psi_0^t \otimes \delta_\emptyset,$$

both right-hand sides of these equations coincide on future vacuum δ_ϕ if $\hat{\psi}^t = \Psi_0^t$ as

$$\mathrm{L}^\kappa\,\mathrm{d}\hat{w}_\kappa^t\,\hat{\psi}(t) = (\mathrm{L}^\kappa\,\mathrm{d}A_\kappa^+ + \mathrm{L}_\kappa^\dagger\,\mathrm{d}A_-^\kappa)(\hat{\psi}^t \otimes \delta_\emptyset) = \mathrm{L}^\kappa \hat{\psi}^t \otimes \mathrm{d}A_\kappa^+ \delta_\emptyset$$

$$\frac{i}{\hbar}\mathrm{L}^\kappa\,\mathrm{d}\,\hat{f}_\kappa^t\,\hat{\psi}_0(t) = (\mathrm{L}^\kappa\,\mathrm{d}A_\kappa^+ - \mathrm{L}_\kappa\,\mathrm{d}A_-^\kappa)(\Psi_0^t \otimes \delta_\emptyset) = \mathrm{L}^\kappa \Psi_0^t \otimes \mathrm{d}A_\kappa^+ \delta_\emptyset$$

(the annihilation processes A_-^κ are zero on the vacuum $\delta_\emptyset$). By virtue of the coincidence of the intial data $\hat{\psi}^0 = \psi = \Psi_0^t$ this proves that $\hat{\psi}(t) = \Psi_0(t)$ for all $t > 0$. Note that the quantum stochastic evolutions $\hat{\psi}(t)$ and $\Psi_0(t)$ when extended on the whole space $\mathfrak{h} \otimes \mathcal{G}_0$, are described by the different propagators $V(t)$ and $W(t)$ as $\hat{\psi}(t) = V(t)\hat{\psi}$, $\Psi_0(t) = W(t)\Psi_0$. The first one is unbounded and even not well-defined on the whole space $\mathfrak{h} \otimes \mathcal{G}$, while the second one is unitary, resolving another stochastic differential equation

$$\mathrm{d}\psi_0(t) + \left(\frac{i}{\hbar}\mathrm{H} + \frac{1}{2}\mathrm{Q}^\kappa \lambda_\kappa^2 \mathrm{Q}^\kappa\right)\psi_0(t)\,\mathrm{d}t = \frac{i}{\hbar}\lambda \mathrm{Q}^\kappa \psi_0(t)\,\mathrm{d}f_\kappa^t, \quad \psi_0(0) = \psi, \quad (6)$$

by the unitary propagator $W(t, f) = W(t)(\mathrm{I} \otimes \delta_\phi)$ for each f in $\mathfrak{h}$ as the stochastic function $\psi_0(t, f) = W(t, f)\psi$ on another classical probability space.

Thus the stochastic decoherence equation,

$$\mathrm{d}\psi(t) + \left(\frac{i}{\hbar}\mathrm{H} + \frac{1}{2}\mathrm{Q}^\kappa \lambda_\kappa^2 \mathrm{Q}^\kappa\right)\psi(t)\,\mathrm{d}t = \lambda_\kappa \mathrm{Q}^\kappa \psi(t)\,\mathrm{d}w_\kappa^t, \quad \psi(0) = \psi,$$

for the continuous observation of the position of a quantum particle with $\mathrm{H} = \frac{1}{2m}\mathrm{P}^2 + \phi(Q)$ was derived for the unitary quantum stochastic evolution as an example of the general decoherence equation which was obtained in this way in Belavkin (1988b). It was explicitly solved in the literature (Belavkin, 1988a, 1989; Belavkin and Staszewski, 1992) for the case of linear and quadratic potentials ϕ, and it was shown that this solution coincides with the optimal quantum linear filtering solution obtained earlier in (Belavkin, 1980, 1985) if the initial wave packet is Gaussian.

The nonlinear stochastic posterior equation for this particular case was derived independently by Diosi (1988) and (as an example) in Belavkin (1988a, 1989). It has the follwing form

$$\mathrm{d}\psi_w(t) + \left(\frac{i}{\hbar}\mathrm{H} + \frac{1}{2}\tilde{\mathrm{Q}}^\kappa(t)\lambda_\kappa^2 \tilde{\mathrm{Q}}^\kappa(t)\right)\psi_w(t)\,\mathrm{d}t = \lambda_\kappa \tilde{\mathrm{Q}}^\kappa(t)\,\psi_w(t)\,\mathrm{d}\tilde{w}_\kappa^t,$$

where $\tilde{\mathrm{Q}}(t) = \mathrm{Q} - \hat{q}(t)$ with $\hat{q}^\kappa(t)$ defined as the multiplication operators by the components $q^\kappa(t, w) = \psi_w^\dagger(t)\,\mathrm{Q}^\kappa(t)\psi_w(t)$ of the posterior expectation (statistical prediction) of the coordinate Q, and

$$\mathrm{d}\tilde{w}_\kappa^t = \mathrm{d}w_\kappa^t - 2\lambda_\kappa \hat{q}^\kappa(t)\,\mathrm{d}t = \mathrm{d}y_\kappa^t - \hat{x}_\kappa(t)\,\mathrm{d}t, \quad \hat{x}_\kappa(t) = 2(\lambda\hat{q})^\kappa(t).$$

Note that the innovating output processes $\tilde{w}_\kappa^t$ are also standard Wiener processes with respect to the output probability measure $\mathrm{d}\tilde{\mu} = \lim_{t\nearrow\infty} \Pr(t, \mathrm{d}\omega)$ but not with respect to the Wiener probability measure $\mu = \Pr(0, \mathrm{d}\omega)$ for the input noise w_κ^t.

Let us give the explicit solution of this stochastic wave equation for the free particle ($\phi = 0$) in one dimension and the stationary Gaussian initial wave packet which was found in the literature (Belavkin, 1988a, 1989; Belavkin and Staszewski, 1992). One can show (Chruscinski and Staszewski, 1992; Kolokoltsov, 1995) that the nondemolition observation of such particle is described by filtering of quantum noise which results in the continual collapse of any wave packet to the Gaussian stationary one centered at the posterior expectation $q(t, w)$ with finite dispersion $\|(\hat{q}(t) - \mathrm{Q})\psi_\omega(t)\|^2 \to 2\lambda(\hbar/m)^{1/2}$. This center can be found from the linear Newton equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + 2\kappa \frac{\mathrm{d}}{\mathrm{d}t}z(t) + 2\kappa^2 z(t) = -g(t),$$

for the deviation process $z(t) = q(t) - x(t)$, where $x(t)$ is an expected trajectory of the output process (1) with $z(0) = q_0 - x(0)$, $z'(0) = v_0 - x'(0)$. Here $\kappa = \lambda(\hbar/m)^{1/2}$ is the decay rate which is also the frequency of effective oscillations, $q_0 = \langle\hat{x}\rangle$, $v_0 = \langle\hat{p}/m\rangle$ are the initial expectations and $g(t) = x''(t)$ is the effective gravitation for the particle in the moving framework of $x(t)$. The solution to the above equation illustrates the continuous collapse $z(t) \to 0$ of the posterior trajectory $q(t)$ towards a linear trajectory $x(t)$. The posterior position expectation $q(t)$ in the absence of effective gravitation, $x''(t) = 0$, for the linear trajectory $x(t) = ut - q$ collapses to the expected input trajectory $x(t)$ with the rate $\kappa = \lambda(\hbar/m)^{1/2}$, remaining not collapsed, $q_0(t) = v_0(t)$ in the framework where $q_0 = 0$, only in the classical limit $\hbar/m \to 0$ or absence of observation $\lambda = 0$. This is the graph of

$$q_0(t) = v_0 t, \quad q(t) = ut + e^{-\kappa t}(q \cos\kappa t + (q + \kappa^{-1}(v_0 - u)) \sin\kappa t) - q$$

obtained as $q(t) = x(t) + z(t)$ by explicit solving of the second-order linear equation for $z(t)$.

4. THE EVENTUM MECHANICS REALIZATION

Finally, let us describe the *Eventum Mechanics* underlying all quantum diffusion and more general quantum noise Langevin models of information dynamics. We shall see that all such phenomenological models exactly correspond to Dirac-type boundary value problems for a Poisson flow of independent quantum particles interacting with the quantum system under the observation at the boundary $r = 0$ of the half line $\mathbb{R}_+$ in an additional dimension. The second-quantized massless Dirac equation (7) with corresponding boundary condition (8), together with the quantum causality (or nondemolition) superselection rule, is the essence of the Eventum Mechanics, the new, extended quantum mechanics in which there is a place for the phenomenological events such as quantum trajectories and spontaneous localizations. One can think of the coordinate $r > 0$ being perpendicular to the quantum target membrane of a scattering measuring device, or an extra dimension coordinate as a physical realization of localizable time in our m-brane universe. At least it is so for any free evolution Hamiltonian $\varepsilon(p) > 0$ of the incoming quantum particles in the ultrarelativistic limit $\langle p\rangle \to -\infty$ such that the average velocity in an initial state is a finite constant, $c = \langle\varepsilon'(p)\rangle \to 1$ say, see in details on this limit of an idealized rigid boundary measurement schemes in the literature (Belavkin, 2000, 2001; Belavkin and Kolokoltsov, 2001). Thus we are going to solve the microscopic foundation problem for quantum trajectories, individual decoherence, state diffusion, or permanent reduction theories as the following time-continuous information dynamics derivation problem:

> Let $V(t,\omega) = V(t,\omega_0^{t]})$, $t \in \mathbb{R}_+$ be a reduction family of isometries on $\mathfrak{h}$ into $\mathfrak{h} \otimes L^2_\mu$ resolving the state diffusion Eq. (5) with respect to the input probability measure $\mu = \mathsf{P}_w$ for the standard Wiener noises w^t_κ, defining the classical means
>
> $$\mathsf{M}[gV(t)^\dagger \mathrm{B}V(t)] = \int g\left(w_0^{t]}\right)V\left(t, w_0^{t]}\right)^\dagger \mathrm{B}V\left(w_0^{t]}\right)\mathrm{d}\mathsf{P}_w.$$
>
> Find a triple $(\mathcal{G}, \mathfrak{A}, \Phi)$ consisting of a Hilbert space $\mathcal{G} = \mathcal{G}_- \otimes \mathcal{G}_+$ embedding the Wiener–Hilbert space L^2_μ by an isometry into $\mathcal{G}_+$, an algebra $\mathfrak{A} = \mathfrak{A}_- \otimes \mathfrak{A}_+$ on $\mathcal{G}$ with an Abelian subalgebra $\mathfrak{A}_-$ generated by a compatible continuous family $Y^{o]}_{-\infty} = \{Y^s_\kappa, \kappa = 1, \ldots, d, s \leq 0\}$ of observables on $\mathcal{G}_-$, and a state-vector $\Phi^\circ = \Phi^\circ_- \otimes \Phi^\circ_+ \in \mathcal{G}$ such that there exists a time-continuous unitary group U^t on $\mathcal{H} = \mathfrak{h} \otimes \mathcal{G}$, inducing a semigroup of endomorphisms $\mathfrak{B} \ni B \mapsto U^{-t}BU^t \in \mathfrak{B}$, which represents this reduction on the product algebra $\mathfrak{B} = B(\mathfrak{h}) \otimes \mathfrak{A}$ as
>
> $$\mathsf{M}[gV(t)^\dagger \mathrm{B}V(t)] = \pi^t(\hat{g}_{-t} \otimes B).$$
>
> Here π^t is the quantum conditional expectation
>
> $$\pi^t(\hat{g}_{-t} \otimes \mathrm{B}) = (\mathrm{I} \otimes \Phi^\circ)^\dagger U^{-t}\left(\mathrm{B} \otimes g_{-t}\left(Y^{0]}_{-t}\right)\right)U^t(\mathrm{I} \otimes \Phi^\circ),$$
>
> which provides the dynamical realization of the reduction as the statistically causal inference about any $\mathrm{B} \in \mathcal{B}(\mathfrak{h})$ with respect to the algebra $\mathfrak{A}_-$ of the functionals $\hat{g}_{-t} = g_{-t}(Y^{0]}_{-t})$, of $Y^{-0]}_{-t} = \{Y^s_\cdot : s \in (-t, 0]\}$, all commuting on $\mathcal{G}$, representing the shifted measurable functionals $g_{-t}(y^{0]}_{-t}) = g(y^{t]}_0)$ of $y^{t]}_0 = \{y^r_\cdot : r \in (0,t]\}$ for each $t > 0$ in the center $\mathfrak{C}$ of the algebra $\mathfrak{B}$.

We have already dilated the state diffusion Eq. (5) to a quantum stochastic unitary evolution $W(t)$ resolving the quantum stochastic Schrödinger Eq. (3) on the system Hilbert $\mathfrak{h}$ tensored with the Fock space $\mathcal{F}_0$ such that $W(t)(\mathrm{I} \otimes \delta_\emptyset) = V(t)$, where $\delta_\phi \in \mathcal{F}_0$ is the Fock vacuum vector. In fact the state diffusion equation was first derived (Belavkin, 1988a, 1989) in this way from even more general quantum stochastic unitary evolution which satisfies the equation

$$(\mathrm{I} \otimes \delta_\emptyset)^\dagger W(t)^\dagger\left(\mathrm{B} \otimes g\left(\hat{w}_0^{t]}\right)\right)W(t)(\mathrm{I} \otimes \delta_\phi) = \mathsf{M}[gV(t)^\dagger BV(t)].$$

Indeed, this equation is satisfied for the model (3) as one can easily check for

$$g\left(w_0^{t]}\right) = \exp\left[\int_0^t \left(f^\kappa(r)\,\mathrm{d}w^r_\kappa - \frac{1}{2}f(r)^2\,\mathrm{d}r\right)\right]$$

given by a test vector function f by conditioning the Langevin equation (4) with respect to the vacuum vector $\delta_\emptyset$:

$$(\mathrm{I} \otimes \delta_\phi)^\dagger(\mathrm{d}X + (K^\dagger X + XK - L^\dagger XL - (XL + L^\dagger X)\,f)\,\mathrm{d}t)(\mathrm{I} \otimes \delta_\phi) = 0.$$

Obviously this equation coincides with the conditional expectation

$$\mathsf{M}[\mathrm{d}B(t) + (\mathrm{K}^\dagger B(t) + B(t)\mathrm{K} - \mathrm{L}^\dagger B(t)\mathrm{L} - (B(t)L + L^\dagger B(t))\,f(t))\,\mathrm{d}t] = 0$$

for the stochastic process $B(t) = V(t)^{\dagger} gXV(t)$ which satisfies the stochastic Itô equation

$$\mathrm{d}B + gV^{\dagger}(\mathrm{K}^{\dagger}\mathrm{X} + \mathrm{XK} - \mathrm{L}^{\dagger}\mathrm{XL} - (\mathrm{XL} + \mathrm{L}^{\dagger}\mathrm{X})f)V\,\mathrm{d}t$$
$$= gV^{\dagger}(\mathrm{L}^{\dagger}\mathrm{X} + \mathrm{XL} + \mathrm{X})V\,\mathrm{d}w^{t}.$$

This however does not give yet the complete solution of the quantum measurement problem as formulated above because the algebra $\mathfrak{B}_0$ generated by $B \otimes I$ and the Langevin forces $\hat{f}^{t}_{\kappa}$ does not contain the measurement process $\hat{w}^{t}_{\kappa}$ which do not commute with $\hat{f}^{t}_{\kappa}$, and the unitary family $W(t)$ does not form unitary group but only cocycle

$$T_t W(s) T_{-t} W(t) = W(s+t), \qquad \forall_s, t > 0,$$

with respect to the isometric but not unitary right shift semigroup T_t in $\mathcal{F}_0$.

Let T_t be the one parametric continuous unitary shift group on $\mathcal{F}^{o]} \otimes \mathcal{F}_0$ extending the definition from $\mathcal{F}_0$. It describes the free evolution by right shifts $\Phi_t(\omega) = \Phi(\omega - t)$ in Fock space over the whole line $\mathbb{R}$. Then one can easily find the unitary group

$$U^{t} = T_{-t}(I^{0} \otimes \mathrm{I} \otimes W(t))T_t$$

on $\mathcal{F}_0 \otimes \mathfrak{h} \otimes \mathcal{F}^{0]}$ inducing the quantum stochastic evolution as the interaction representation $U(t) = T_t U^{t}$ on the Hilbert space $\mathfrak{h} \otimes \mathcal{G}_0$. In fact this evolution corresponds to an unphysical coordinate discontinuity problem at the origin $s = 0$ which is not invariant under the reflection of time $t \mapsto -t$. Instead, we shall formulate the unitary equivalent boundary value problem in the Poisson space $\mathcal{G} = \mathcal{G}_{-} \otimes \mathcal{G}_{+}$ for two semi-infinite strings on $\mathbb{R}_{+}$, one is the living place for the quantum noise generated by a Poisson flow of incoming waves of quantum particles of the intensity $\nu > 0$, and the other one is for the outgoing classical particles carrying the information after a unitary interaction with the measured quantum system at the origin $r = 0$. The probability amplitudes $\Phi \in \mathcal{G}$ are represented by the $\mathbb{G}^{\otimes} = \mathrm{g}_{-}^{\otimes} \otimes \mathrm{g}_{+}^{\otimes}$ valued functions $\Phi(v_{-}, v_{+})$ of two infinite sequences $v_{\pm} = \{\pm r_1, \pm r_2, \ldots\} \subset \mathbb{R}_{+}$ of the coordinates of the particles in the increasing order $r_1 < r_2 < \cdots$ such that

$$\|\Phi\|^2 = \iint \|\Phi(v_{-}, v_{+})\|^2\,\mathrm{P}_{\nu}\,(\mathrm{d}v_{-}) < \infty$$

with respect to the product of two copies of the Poisson probability measure P_{ν} defined by the constant intensity $\nu > 0$ on $\mathbb{R}_{+}$. Here $\mathrm{g}^{\otimes}$ is the infinite tensor product of $\mathrm{g} = \mathbb{C}^{d}$ obtained by the completion of the linear span of $\chi_1 \otimes \chi_2 \otimes \cdots$ with almost all multipliers $\chi_n = \varphi$ given by a unit vector $\varphi \in \mathbb{C}^{d}$ such that the infinite product $\|\Phi(v)\| = \prod_{r \in v} \|f(r)\|$ for $\Phi(v) = \otimes_{r \in v} f(r)$ with $f(r_n) = \chi_n$ is well defined as it has all but finite number of multipliers $\|\chi_n\|$ equal 1. The unitary transformation $F \mapsto \Phi$ from a Fock space $\mathcal{F} \ni F$ to the corresponding Poisson one $\mathcal{G}$ can be written as

$$\Phi = \lim_{t\to\infty} e^{\varphi A_i^+(t)} e^{-\varphi\kappa A_-^\kappa(t)} \nu^{-\frac{1}{2}A_i^i(t)} F \equiv I_\nu(\varphi)F,$$

where $\varphi_\kappa = \nu_\varphi^{-\kappa}$ for the Poisson intensity $\nu > 0$ and the unit vector $\varphi = (\varphi^i)$ defined by the initial probability amplitude $\varphi \in \mathfrak{g}$ for the auxiliary particles to be in a state $\kappa = 1, \dots, d$. Here $A_i^\kappa(t)$ are the QS integrators defined in the Appendix, and the limit is taken on the dense subspace $\cup_{t>0}\mathcal{F}_0^{t]}$ of vacuum-adapted Fock functions $F_t \in \mathcal{F}_0$ and extended then onto $\mathcal{F}_0$ by easily proved isometry $\|F_t\| = \|\Phi_t\|$ for $\Phi_t = I_\nu(\varphi)F_t$.

The free evolution is $\mathcal{G}$ is the left shift for the incoming waves and the right shift for outgoing waves,

$$T_t\Phi(\nu_-, \nu_+) = \Phi(\nu_-^t, \nu_+^t),$$

where $\nu_\pm^t = \pm[([(-\nu_-) \cup (+\nu_+)] - t) \cap \mathbb{R}_\pm]$. It is given by the second quantization

$$\mathrm{P}\Phi(\nu_-, \nu_+) = \frac{\hbar}{\mathrm{i}}\left(\sum_{r\in\nu_+}\frac{\partial}{\partial r} - \sum_{r\in\nu_-}\frac{\partial}{\partial r}\right)\Phi(\nu_-, \nu_+)$$

of the Dirac–Hamiltonian in one dimension on $\mathbb{R}_+$.

To formulate the boundary value problem in the space $\mathcal{H} = \mathfrak{h} \otimes \mathcal{G}$ corresponding to the quantum stochastic equations of the diffusive type (3), let us introduce the notation

$$\Phi\,(0^\kappa \cup \nu_\pm) = \lim_{r\searrow 0}(\langle\kappa| \otimes \mathrm{I}_1 \otimes \mathrm{I}_2 \dots)\Phi(\pm r, \pm r_1, \pm r_2, \dots),$$

where $\langle\kappa| = d^{-1/2}(\delta_1^\kappa, \dots, \delta_l^\kappa)$ acts as the unit bra-vector evaluating the kth projection of the state vector $\Phi(\pm r \sqcup \nu_\pm)$ with $r < r_1 < r_2 < \dots$ corresponding to the nearest to the boundary $r = 0$ particle in one of the strings on $\mathbb{R}_+$.

The unitary group evolution U_t corresponding to the scattering interaction at the boundary with the continuously measured system which has its own free evolution described by the energy operator $\mathrm{E} = \mathrm{E}^\dagger$ can be obtained by resolving the following generalized Schrödinger equation,

$$\frac{\partial}{\partial t}\Psi^t(\nu_-, \nu_+) = \frac{i}{\hbar}\mathrm{P}\Psi^t(t, \nu_-, \nu_+) + \mathrm{G}_+^-\Psi^t(\nu_-, \nu_+) + \mathrm{G}_\kappa^-\Psi^t(\nu_-, 0^k \sqcup \nu_+), \tag{7}$$

with the Dirac zero current boundary condition at the origin $r = 0$,

$$\Psi^t(0^i \sqcup \nu_-, \nu_+) = \mathrm{G}_+^i\Psi^t(\nu_-, \nu_+) + \mathrm{G}_\kappa^i\Psi^t(\nu_-, 0^k \sqcup \nu_+), \quad \forall t > 0,\ \nu_\pm > 0. \tag{8}$$

Here $G = [\mathrm{G}^i_\kappa]$ is unitary, $\mathrm{G}^{-1} = \mathrm{G}^\dagger$, the scattering operator S in the simpler quantum jump boundary value problem corresponding to $\mathrm{G}_+ = 0 = \mathrm{G}^-$, and the other system operators G^κ_ι, with $\iota = -, i$ and $\kappa = k, +$ for any $i, k = 1, \ldots, d$ are chosen as

$$\mathrm{G}^- + \nu \mathrm{G}^\dagger_+ \mathrm{G} = 0, \qquad \mathrm{G}^-_+ + \frac{\nu}{2}\mathrm{G}^\dagger_+ \mathrm{G}_+ + \frac{i}{\mathrm{E}} = \mathrm{O}. \tag{9}$$

Note that these conditions can be written as pseudounitarity operator of the following triangular block-matrix:

$$\begin{bmatrix} \mathrm{I} & \mathrm{G}^- & \mathrm{G}^-_+ \\ \mathrm{O} & \mathrm{G} & \mathrm{G}^+ \\ \mathrm{O} & \mathrm{O} & \mathrm{I} \end{bmatrix}^{-1} = \begin{bmatrix} \mathrm{O} & \mathrm{O} & \mathrm{I} \\ \mathrm{O} & \nu\mathrm{I} & \mathrm{O} \\ \mathrm{I} & \mathrm{O} & \mathrm{O} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{I} & \mathrm{G}^- & \mathrm{G}^-_+ \\ \mathrm{O} & \mathrm{G} & \mathrm{G}^+ \\ \mathrm{O} & \mathrm{O} & \mathrm{I} \end{bmatrix}^{\dagger} \begin{bmatrix} \mathrm{O} & \mathrm{O}^- & \mathrm{I} \\ \mathrm{O} & \nu\mathrm{I} & \mathrm{O} \\ 1 & \mathrm{O} & \mathrm{O} \end{bmatrix}.$$

As it was proved in Belavkin (1988b, 1992a) this is a necessary (and sufficient if all operators are bounded) condition for the unitarity $W(t)^{-1} = W(t)^\dagger$ of the cocycle solution resolving the quantum stochastic differential equation

$$\mathrm{d}\Psi_0(t) = \left(\mathrm{G}^i_\kappa - \delta^i_\kappa \mathrm{I}\right) \Psi_0(t)\, \mathrm{d}A^\kappa_i, \quad \Psi_0(0) = \Psi_0$$

in the Hilbert space $\mathcal{H}_0 = \mathfrak{h} \otimes \mathcal{G}_0$ where $\mathcal{G}_0$ is identified with the space $\mathcal{G}_+ = \mathbb{G}^\otimes \otimes L^2_\mu$ for the Poisson measure $\mu = \mathrm{P}_\nu$ with the intensity ν on $\mathbb{R}_+$. This is the general form for the quantum stochastic Eq. (3), where $\mathrm{d}A^+_- = \mathrm{d}t$ in the Poisson space (see the Appendix for more detailed explanations of these notations). Our recent results partially published in the literature (Belavkin, 2000a,b, 2001; Belavkin and Kolokoltsov, 2001) prove that this quantum stochastic evolution extended as the identity I_- also on the component $\mathcal{G}_-$ for the scattered particles is nothing but the interaction representation $U^t = T_{-t}(I_- \otimes W(t))$ for the unitary group U^t resolving our boundary value problem in $\mathfrak{h} \otimes \mathbb{G}^\otimes$ times the Poisson space L^2_μ. Thus the pseudounitarity condition (9) is necessary (and sufficient if the operators G^i_κ are bounded) for the self-adjointness of the Dirac-type boundary value problem (7) and (8).

The generators G^ι_κ of this boundary value problem define the generators S^ι_κ of the corresponding quantum stochastic equation in Fock space by the following transformation,

$$\mathrm{S}^i_+ = \nu^{1/2}(\mathrm{G}^i_+ + \mathrm{G}^i{}_\kappa \varphi^\kappa - \varphi^i),\, \mathrm{S}^-_\kappa = \nu^{-1/2}\left(\mathrm{G}^i_\kappa + \varphi_i \mathrm{G}^i_\kappa - \varphi_\kappa\right)$$
$$\mathrm{S}^-_+ = \mathrm{G}^-_+ + \varphi_i \mathrm{G}^i_+ + \mathrm{G}^-_\kappa \varphi^\kappa + \varphi_i\left(\mathrm{G}^i_\kappa - \delta^i_\kappa \mathrm{I}\right)\varphi^\kappa, \quad \mathrm{S}^i_\kappa = \mathrm{G}^i_\kappa.$$

induced by the canonical transformation $I_\nu(\varphi)$.

The quantum state diffusion equation (5) for the continuous measurement of the coordinates Q^k corresponds to the particular case (2) of the quantum stochastic differential equation in Fock space, with

$$S^i_+ = \nu^{1/2} G^i_+, \qquad S^-_k = \nu^{-1/2} G^-_k$$
$$S^-_+ = G^-_+ + \varphi_i G^i_+ + G^-_k \varphi^k, \qquad S^i_k = \delta^i_k I,$$

and $G^i_+ = Q^i$, $G^-_k = \nu Q^k$ such that all coupling constants $\lambda_k = \nu^{1/2}$ are equal to the square root of the flow intensity ν. The operators $G^i_+ = \varphi^i Q^i$ and $G^-_k = Q_k \varphi_k$ corresponding to the different couplings λ_k can also be obtained from the purely jump model in the central limit $\nu \mapsto \infty$ as it was done in Belavkin and Melsheimer (1996). In this case

$$S^i_+ = \nu^{1/2}(G - I)^i_k \varphi^k \to -i\lambda\varphi^i Q^i,$$

with $\varphi^k = \mathrm{i}\lambda_k/\lambda$.

And finally, we have to find the operator processes Y^s_k, $s \le 0$, on the Hilbert space $\mathcal{G}_-$ which reproduce the standard Wiener noises w^t_k in the state diffusion when our dynamical model is conditioned (filtered) with respect to their nondemolition measurement. As the candidates, let us consider the field coordinate processes

$$X^{-t}_k = A^+_k(-t, 0] + A^k_-(-t, 0] = T_{-t}(A^+_k(0, t] + A^k_-(0, t])\, Tt,$$

which are given by the creation and annihilation processes $A^+(t)$ and $A_-(t)$ shifted from $\mathcal{G}_+$. In our Poisson space model of $\mathcal{G}$ they do not have zero expectations

$$\Phi^\dagger X^{-t}_k \Phi = \Phi^\dagger(A^+_k(0, t] + A^k_-(0, t])\Phi = 2\nu^{1/2} t$$

in the ground state $\Phi = I_\nu(\varphi)\delta_\emptyset$ corresponding to the vacuum vector $\delta_\emptyset$ in the Fock space. This state is given as the infinite tensor product $\Phi^{\mathrm{o}} = \varphi^\otimes_- \otimes \varphi^\otimes_+$ of all equal probability amplitudes $\varphi_- = \varphi = \varphi_+$ in $g = \mathbb{C}^d$ for each sequence v_- and v_+. Hence the independent increment processes $Y^t_k = T_t Y^{-t}_k T_{-t}$ corresponding to the standard Wiener noises w^t_k represented in Fock spaces as $\hat{w}^t_k = A^+_k(t) + A^k_-(t)$ are the compensated processes $Y^{-t}_k = X^{-t}_k - 2\nu^{1/2} t$. This unitary equivalence of Y^t_k and $\hat{w}^t_k$ under the Fock–Poisson transformation $I_\nu(\varphi)$, and the deduction given above of the quantum state diffusion from the quantum stochastic signal plus noise model (1) for continuous observation in Fock space, completes the solution of the quantum measurement model in its rigorous formulation.

5. CONCLUSION: A QUANTUM MESSAGE FROM THE FUTURE

Recent phenomenological theories of continuous reduction, quantum state diffusion, and quantum trajecories extended the instantaneous projection postulate to a certain class of continuous-in-time measurements. As was shown here, there is no need to supplement the usual quantum mechanics with any of such generalized reduction postulate even in the continuous time. They all have been derived from the time-continuous unitary evolution for a generalized Dirac-type

Schrödinger equation with a singular scattering interaction at the boundary of our Hamiltonian model (see the recent review paper, Belavkin (2002)). The quantum causality as a new superselection rule provides a time-continuous nondemolition measurement in the extended system which enables to obtain the quantum state diffusion and quantum trajectories simply by time-continuous conditioning called quantum filtering. Our nondemolition causality principle, which was explicitly formulated in (Belavkin, 1994), admits to select a continuous diffusive classical process in the quantum extended world which satisfies the nondemolition condition with respect to all future of the measured system. And this allows us to obtain the continuous trajectories for quantum state diffusion by simple filtering of quantum noise exactly as it was done in the classical statistical nonlinear filtering and prediction theory. In this way we derived the quantum state diffusion of a Gaussian wave packet already in Belavkin (1979, 1980) as the result of the solution of quantum prediction problem by filtering the quantum white noise in a quantum stochastic Langevin model for the continuous observation. Thus the "primary" for the conventional quantum mechanics stochstic nonlinear irreversible quantum state diffusion appears to be the secondary, as it should be, to the deterministic linear unitary reversible evolution of the extended quantum mechanics containing necessarily infinite number of auxiliary particles. However quantum causality, which defines the arrow of time by selecting what part of the reversible world is related to the classical past and what is related to the quantum future, makes the extended mechanics irreversible in terms of the injective semigroup of the invertible Heisenberg transformations indeuced by the unitary group evolution for the positive arrow of time. The microscopic information dynamics of this event enhanced quantum mechanics, or Eventum Mechanics, allows the emergence of the decoherence and the increase of entropy in a purely dynamical way without any sort of reservoir averaging.

Summarizing, we can formulate the general principles of the Eventum Mechanics which unifies the classical and quantum mechanics in such a way that there is no contradiction between the unitary evolution of the matter waves and the phenomenological information dynamics such as quantum state diffusion or spontaneous jumps for the events and the trajectories of the particles. This can be described as the conventional, *nonstochastic* but time-asymmetric quantum mechanics, with different linear operators for the observables and evolution generators in the Hilbert space, which is minimally extended in the following way:

- It is a reversible wave mechanics of the continuous unitary group evolutions in an infinite-dimensional Hilbert space.
- It has conventional interpretation for the normalized Hilbert space vectors as state vectors (probability amplitudes).
- However not all operators, e.g., the dynamical generator (Hamiltonian), are admissible as the potential observables.

– Quantum causality is statistical predictability of the quantum states based on the results of the actual measurements.
– It implies the choice of time arrow and an initial state which, together with past measurement data, defines the reality.
– The actual observables (beables) must be compatible with any operator representing a potential (future) observable.
– The Heisenberg dynamics and others symmetries induced by unitary operators should be algebraically endomorphic.
– However these endomorphisms form only a semigroup on the algebra of all observables as they may be irreversible.

Note that the classical Hamiltonian mechanics can be also described in this way by considering only the commutative algebras of the potential observables. Each such observable is compatible with any other and can be considered as an actual observable, or beable. However, the Hamiltonian operator, generating a nontrivial Liouville unitary dynamics in the corresponding Hilbert space, is not an observable, as it does not commute with any observable which is not the integral of motion. Nevertheless the corresponding Heisenberg dynamics, described by the induced automorphisms of the commutative algebra, is reversible, and pure states, describing the reality, remain pure, nondisturbed by the measurements of its observables. This is also true in the purely quantum mechanical case, in which the Hamiltonian is an observable, as there are no events and nontrivial beables in the conventional quantum mechanics. The only actual observables, which are compatible with any Hermitian operator as a potential observable, are the constants, i.e., proportional to the identity operator, as the only operators, commuting with any such observable. Their measurements do not bring new information and do not disurb the quantum states. However any nontrivial classical–quantum Hamiltonian interactions cannot induce a group of the reversible Heisenberg automorphisms but only a semigroup of irreversible endomorphisms of the decomposable algebra of all potential observables of the composed classical-quantum system. This follows from the simple fact that any automorphism leaves the center of an operator algebra invariant, and thus induces the autonomous noninteracting dynamics on the classical part of the semiclassical system. This is the only reason which is responsible for failure of all earlier desperate attempts to build the reversible, time-symmetric Hamiltonian theory of classical-quantum interaction which would give a dynamical solution of the quantum decoherence and measurement problem along the line suggested by von Neumann and Bohr. There is no nontrivial reversible classical-quantum mechanical interaction, but as we have seen, there is a Hamiltonian irreversible interaction within the time-asymmetric Eventum Mechanics.

The unitary solution of the described boundary value problem indeed induces endomorphic semiclassical Hamiltonian dynamics, and in fact is underlying in any phenomenological reduction model (Belavkin, 2002). Note that although the irreversible Heisenberg endomorphisms of eventum mechanics, induced by the unitary propagators, are injective, and thus are invertible by completely positive maps, and are not mixed, they mix the pure states over the center of the algebra. Such mixed states, which are uniquely represented as the orthogonal mixture over the "hidden" variables (beables), can be filtered by the measurement of the actual observables, and this transition from the prior state corresponding to the less definite (mixed) reality to the posterior state corresponding to a more definite (pure) reality by the simple inference does not change the reality. This is an explanation, in the pure dynamical terms of the eventum mechanics, of the emergence of the decoherence and the reductions due to the measurement, which has no explanation in the conventional classical and quantum mechanics.

Our mathematical formulation of the eventum mechanics as the extended quantum mechanics equipped with the quantum causality to allow events and trajectories in the theory, is just as continuous as Schrödinger could have wished. However, it does not exclude the jumps which only appear in the singular interaction picture, which are there as a part of the theory, not only of its interpretation. Although Schrödinger himself did not believe in quantum jumps, he tried several times, although unsuccessfully, to obtain the continuous reduction from a generalized, relativistic, "true Schrödinger" equation. He envisaged that "if one introduces two symmetric systems of waves, which are traveling in opposite directions; one of them presumably has something to do with the known (or supposed to be known) state of the system at a later point in time" (Shrödinger, 1931), then it would be possible to derive the "verdammte Quantenspringerei" for the opposite wave as a solution of the future-past boundary value problem. This desire coincides with the "transactional" attempt of interpretation of quantum mechanics suggested in Cramer (1986) on the basis that the relativistic wave equation yields in the nonrelativistic limit two Schrödinger-type equations, one of which is the time-reversed version of the usual equation:

> The state vector ψ of the quantum mechanical formalism is a real physical wave with spatial extension and it is identical with the initial "offer wave" of the transaction. The particle (photon, electron, etc.) and the collapsed state vector are identical with the completed transaction.

There was no proof of this conjecture, and now we know that it is not even possible to derive the quantum state diffusions, spontaneous jumps, and single reductions from models involving only a finite particle state vectors $\psi(t)$ satisfying the conventional Schrödinger equation.

Our new approach, based on the exactly solvable boundary value problems for infinite particle states described in this paper, resolves the problem formulated by Schrödinger. And thus it resolves the old problem of interpretation of the quantum theory, together with its infamous paradoxes, in a constructive way by giving exact nontrivial models for allowing the mathematical analysis of quantum observation processes determining the phenomenological coupling constants and the reality underlying these paradoxes. Conceptually it is based upon a new idea of quantum causality called the nondemolition principle (Belavkin, 1994) which divides the world into the classical past, forming the consistent histories, and the quantum future, the state of which is predictable for each such history.

6. APPENDIX: SYMBOLIC QUANTUM CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS

To formulate the differential nondemolition causality condition and to derive a filtering equation for the posterior states in the time-continuous case, we need quantum stochastic calculus.

The classical differential calculus for the infinitesimal increments

$$dx = x(t + dt) - x(t)$$

became generally accepted only after Newton gave a simple algebraic rule $(\mathrm{d}t)^2 = 0$ for the formal computations of the differentials $\mathrm{d}x$ for smooth trajectories $t \mapsto x(t)$. In the complex plane $\mathbb{C}$ of phase space it can be represented by a one-dimensional algebra $\alpha = \mathbb{C}\mathrm{d}_t$ of the elements $a = \alpha \mathrm{d}_t$ with involution $a^\star = \bar{\alpha}\mathrm{d}_t$. Here

$$\mathrm{d}_t = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y)$$

for $\mathrm{d}t$ is the nilpotent matrix, which can be regarded as Hermitian $\mathrm{d}_t^\star = \mathrm{d}_t$ with respect to the Minkowski metrics $(\mathbf{z}|\mathbf{z}) = 2\Re z_- \bar{z}_+$ in $\mathbb{C}^2$.

This formal rule was generalized to nonsmooth paths early in the last century in order to include the calculus of forward differentials $\mathrm{d}w \simeq (\mathrm{d}t)^{1/2}$ for continuous diffusions w_t which have no derivative at any t, and the forward differentials $\mathrm{d}n \in \{0, 1\}$ for left continuous counting trajectories n_t which have zero derivative for almost all t (except the points of discontinuity where $\mathrm{d}n = 1$). The first is usually done by adding the rules

$$(\mathrm{d}w)^2 = \mathrm{d}t, \qquad \mathrm{d}w\,\mathrm{d}t = 0 = \mathrm{d}t\,\mathrm{d}w$$

in formal computations of continuous trajectories having the first-order forward differentials $\mathrm{d}x = \alpha \mathrm{d}t + \beta \mathrm{d}w$ with the diffusive part given by the increments of standard Brownian paths w. The second can be done by adding the rules

$$(\mathrm{d}n)^2 = \mathrm{d}n, \qquad \mathrm{d}n\,\mathrm{d}t = 0 = \mathrm{d}t\,\mathrm{d}n$$

in formal computatious of left continuous and smooth for almost all t trajectories having the forward differentials $\mathrm{d}x = \alpha \mathrm{d}t + \gamma \mathrm{d}m$ with jumping part given by the increments of standard compensated Poisson paths $m_t = n_t - t$. These rules were developed by Itô (1951) into the form of a stochastic calculus.

The linear span of dt and dw forms the Wiener-Itô algebra $\mathfrak{b} = \mathbb{C}\mathrm{d}t + \mathbb{C}\mathrm{d}_w$ while the linear span of dt and dn forms the Poisson-Itô algebra $\mathfrak{c} = \mathbb{C}\mathrm{d}_t + \mathbb{C}\mathrm{d}_m$ with the second-order nilpotent $\mathrm{d}_w = \mathrm{d}_w^\star$ and the idempotent $\mathrm{d}_w = \mathrm{d}_w^\star$. They are represented together with d_t by the triangular Hermitian matrices

$$\mathrm{d}_t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d}_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d}_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

on the Minkowski space $\mathbb{C}^3$ with respect to the inner Minkowski product $(\mathbf{z}|\mathbf{z}) = z_- z^- + z_\circ z^\circ + z_+ z^+$, where $z^\mu = \overline{z}_{-\mu}$, $-(-, \circ, +) = (+, \circ, -)$.

Although both algebras $\mathfrak{b}$ and $\mathfrak{c}$ are commutative, the matrix algebra $\mathfrak{a}$ generated by $\mathfrak{b}$ and $\mathfrak{c}$ on $\mathbb{C}^3$ is not

$$\mathrm{d}_w\, \mathrm{d}_m = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{d}_m\, \mathrm{d}_w.$$

The four-dimensional $\star$-algebra $\mathfrak{a} = \mathbb{C}\mathrm{d}_t + \mathbb{C}\mathrm{d}_- + \mathbb{C}\mathrm{d}^+ + \mathbb{C}\mathrm{d}$ of triangular matrices with the canonical basis

$$\mathrm{d}_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

given by the algebraic combinations,

$$\mathrm{d}_- = \mathrm{d}_w \mathrm{d}_m - \mathrm{d}_t, \quad \mathrm{d}^+ = \mathrm{d}_m \mathrm{d}_w - \mathrm{d}_t, \quad \mathrm{d} = \mathrm{d}_m - \mathrm{d}_w,$$

is the canonical representation of the differential $\star$-algebra for one-dimensional vacuum noise in the unified quantum stochastic calculus (Belavkin, 1988b, 1992a). It realizes the HP (Hudson–Parthasarathy) table (Hudson and Parthasarathy, 1984)

$$\mathrm{d}A_- \mathrm{d}A^+ = \mathrm{d}t, \quad \mathrm{d}A_- \mathrm{d}A = \mathrm{d}A_-, \quad \mathrm{d}A \mathrm{d}A^+, \quad (\mathrm{d}A)^2 = \mathrm{d}A,$$

with products equal zero for all other paris, for the multiplication of the canonical counting $\mathrm{d}A = \lambda(\mathrm{d})$, creation $\mathrm{d}A^+ = \lambda(\mathrm{d}^+)$, annihilation $\mathrm{d}A_- = \lambda(\mathrm{d}_-)$, and preservation $\mathrm{d}t = \lambda(\mathrm{d}_t)$ quantum stochastic integrators in Fock space over $L^2(\mathbb{R}_+)$. As was proved recently in Belavkin (1998), any generalized Itô algebra describing a quantum noise can be represented in the canonical way as a $\star$-subalgebra of a quantum vacuum algebra

$$\mathrm{d}A_\mu^\kappa \mathrm{d}A_\iota^\nu = \delta_\iota^\kappa \mathrm{d}A_\mu^\nu, \qquad \iota, \mu \in \{-, 1, \ldots, d\}; \quad \kappa, \nu \in \{1, \ldots, d, +\},$$

in the Fock space with several degrees of freedom d, where $\mathrm{d}A_-^+ = \mathrm{d}t$ and d is restricted by the doubled dimensionality of quantum noise (could be infinite), similar to the representation of every semiclassical system with a given state as a subsystem of quantum system with a pure state. Note that in this quantum Itô product formula $\delta_\kappa^\iota = 0$ if $\iota = +$ or $\kappa = -$ as $\delta_\kappa^\iota \neq 0$ only when $\iota = \kappa$.

The quantum Itô product gives an explicit form

$$\mathrm{d}\psi\,\psi^\dagger + \psi\,\mathrm{d}\psi^\dagger + \mathrm{d}\psi\,\psi^\dagger = \left(\alpha_\kappa^\iota\psi^\dagger + \psi\alpha_\kappa^{\star\iota} + \alpha_j^\iota\alpha_\kappa^{\star\iota}\right)_\kappa^\iota \mathrm{d}A_\kappa^\iota$$

of the term $\mathrm{d}\psi\mathrm{d}\psi^\dagger$ for the adjoint quantum stochastic differentials

$$\mathrm{d}\psi = \alpha_\kappa^\iota\,\mathrm{d}A_\iota^\kappa, \qquad \mathrm{d}\psi^\dagger = \alpha_\kappa^{\star\iota}\,\mathrm{d}A_\iota^\kappa,$$

for evaluation of the product differential

$$\mathrm{d}(\psi\psi^\dagger) = (\psi + \mathrm{d}\psi)(\psi + \mathrm{d}\psi)^\dagger - \psi\psi^\dagger.$$

Here $\alpha_{-\kappa}^{\star\iota} = \alpha_{-\iota}^{\kappa\dagger}$ is the quantum Itô involution with respect to the switch $-(-, +) = (+, -)$, $-(1, \dots, d) = (1, \dots, d)$, introduced in Belavkin (1988b), and the Einstein summation is always understood over $\kappa = 1, \dots, d, +$; $\iota = -1, \dots, d$; and $k = 1, \dots, d$. This is the universal Itô product formula which lies in the heart of the general quantum stochastic calculus (Belavkin, 1988b, 1992a) unifying the Itô classical stochastic calculi with respect to the Wiener and Poisson noises and the quantum differential calculi (Gardiner and Collett, 1985; Hudson and Parthasarathy, 1984) based on the particular types of quantum Itô algebras for the vacuum or finite temperature noises. It was also extended to the form of quantum functional Itô formula and even for the quantum nonadapted case in Belavkin (1991, 1993).

Every stationary classical (real or complex) process x^t, $t > 0$, with $x^0 = 0$ and independent increments $x^{t+\Delta} - x^t$ has mean values $\mathsf{M}[x^t] = \lambda t$. The compensated process $y^t = x^t - \lambda t$, which is called noise, has an operator representation $\hat{x}^t$ in Fock space $\mathcal{F}_0$ the Hilbert space $L^2(\mathbb{R}_+)$ in the form of the integral with respect to basic processes A_j^+, A_-^j, A_k^j such that $\mathsf{F} = f(\hat{x})\delta_\emptyset \simeq f(x)$ in terms of the L_μ^2—Fock isomorphism $f \leftrightarrow \mathsf{F}$ of the chaos expansions

$$f(x) = \sum_{n=0}^{\infty} \int \cdots \int_{0<r_1<\cdots<r_n} \mathsf{F}(r_1, \dots r_n)\,\mathrm{d}y^{r_n} \equiv \int \mathsf{F}(v)\,\mathrm{d}y^v$$

of the stochastic functionals $f \in L_\mu^2$ having the finite second moments $\mathsf{M}[|f|^2] = \|\mathsf{F}\|^2$ and the Fock vectors $\mathsf{F} \in \mathcal{F}_0$. The expectations of the Fock operators $f(\hat{x})$ given by the iterated stochastic integrals f coincides on the vauum state-vector $\delta_\emptyset \in \mathcal{F}$ with their expectation given by the probability measure μ:

$$\mathsf{M}[f(x)] = \langle\delta_\emptyset|\,f(\hat{x})\delta_\emptyset\rangle = \mathsf{F}(\emptyset).$$

If its differential increments $\mathrm{d}x^t$ form a two-dimensional Itô algebra, $\hat{x}^t$ can be represented in the form of a commutative combination of the three basic quantum stochastic increments $A = A_0^0$, $A_- = A_-^0$, $A^+ = A_0^+$. The Itô formula for the process x^t given by the quantum stochastic differential

$$\mathrm{d}\hat{x}^t = \alpha\,\mathrm{d}A + \alpha^-\,\mathrm{d}A_- + \alpha_+\,\mathrm{d}A + \,\mathrm{d}\psi + \alpha_+^-\,\mathrm{d}t$$

can be obtained from the HP product (Hudson and Parthasarathy, 1984)

$$\mathrm{d}\hat{x}^t\mathrm{d}\hat{x}^{t\dagger} = \alpha\alpha^\dagger\,\mathrm{d}A + \alpha^-\alpha^\dagger\,\mathrm{d}A_- + \alpha\alpha^{-\dagger}\,\mathrm{d}A^+ + \alpha^-\alpha^{-\dagger}\,\mathrm{d}t.$$

The noises $y_k^t = x_k^t - \lambda_k t$ with stationary independent increments are called standard if they have the standard variance $\mathsf{M}[(x^t)^2] = t$. In this case

$$\hat{y}_k^t = \left(A_k^+ + A_-^k + \varepsilon_k A_k^k\right)(t) = \varepsilon_k m_k^t + (1-\varepsilon_k)w_k^t,$$

where $\varepsilon_k \geq 0$ is defined by the equation $(\mathrm{d}x_k^t)^2 - \mathrm{d}t = \varepsilon\mathrm{d}x_k^t$. Such, and indeed higher dimensional, quantum noises for continuous measurements in quantum optics were considered in Dunn *et al.* (1992) and Gardiner *et al.* (1992)..

The general form of a quantum stochastic decoherence equation, based on the canonical representation of the arbitrary Itô algebra for a quantum noise in the vacuum of d degrees of freedom, can be written as

$$\mathrm{d}\hat{\psi}(t) = \left(\mathrm{S}_\kappa^\iota - \delta_\kappa^\iota I\right)\mathrm{d}A_\iota^\kappa\hat{\psi}(t), \qquad \hat{\psi}(0) = \psi\otimes\delta_\emptyset, \quad \psi\in\mathfrak{h}.$$

Here L_κ^ι are the operators in the system Hilbert space $\mathfrak{h}\ni\psi$ with $\mathrm{S}_\kappa^{\star-}\mathrm{S}_+^\kappa = 0$ for the mean-square normalization

$$\hat{\psi}(t)^\dagger\hat{\psi}(t) = \mathsf{M}[\psi(t,\cdot)^\dagger\psi(t,\cdot)] = \psi^\dagger\psi$$

with respect to the vacuum of Fock space of the quantum noise, where the Einstein summation is understood over all $\kappa = -, 1, \ldots, d, +$ with the agreement

$$\mathrm{S}_-^- = \mathrm{I} = \mathrm{S}_+^+, \quad \mathrm{S}_-^j = \mathrm{O} = \mathrm{S}_j^+, \qquad j = 1, \ldots, d$$

and $\delta_\kappa^\iota = 1$ for all coinciding $\iota, \kappa \in \{-, 1, \ldots, d, +\}$ such that $\mathrm{L}_\kappa^\iota = \mathrm{S}_\kappa^\iota - \delta_\kappa^\iota = 0$ whenever $\iota = +$ or $\kappa = -$. In the notations $\mathrm{S}_+^j = \mathrm{L}^j$, $\mathrm{S}_+^- = -\mathrm{K}$, $\mathrm{S}_j^- = -\mathrm{K}_j$, $j = 1, \ldots, d$, the decoherence wave equation takes the standard form (Belavkin, 1995, 1997)

$$\mathrm{d}\hat{\psi}(t) + (\mathrm{K}\mathrm{d}t + \mathrm{K}_j\mathrm{d}A_-^j)\hat{\psi}(t) = \left(\mathrm{L}^j\mathrm{d}A_j^+ + \left(\mathrm{S}_\kappa^i - \delta_\kappa^i\mathrm{I}\right)\mathrm{d}A_i^k\right)\hat{\psi}(t),$$

where $A_j^+(t)$, A_-^j, $A_i^k(t)$ are respectively the canonical creation, annihilation and exchange processes in Fock space, and the normalization condition is written as $\mathrm{L}_k\mathrm{L}^k = \mathrm{K} + \mathrm{K}^\dagger$ with $\mathrm{L}_k^\dagger = \mathrm{L}^k$ (the Einstein summation is over $i, j, k = 1, \ldots, d$).

Using the quantum Itô formula, one can obtain the corresponding equation for the quantum stochastic density operator $\hat{\varrho} = \psi\psi^\dagger$ which is the particular case $\kappa = -, 1, \ldots, d, +$ of the general quantum stochastic Master equation

$$\mathrm{d}\hat{\varrho}(t) = \left(\mathrm{S}_\gamma^\iota\hat{\varrho}(t)\mathrm{S}_\kappa^{\star\gamma} - \hat{\varrho}(t)\delta_\kappa^\iota\right)\mathrm{d}A_\iota^\kappa, \qquad \hat{\varrho}(0) = \rho,$$

where the summation over $\kappa = -, k, +$ is extended to infinite number of $k = 1, 2, \ldots$. This general form of the decoherence equation with $\mathrm{L}_\kappa^{\star-}\mathrm{L}_+^\kappa = \mathrm{O}$ corresponding to the normalization condition $\langle\hat{\varrho}(t)\rangle = \mathrm{Tr}\rho$ in the vacuum mean, was recently derived in terms of quantum stochastic completely positive maps in Belavkin (1995, 1997). Denoting $\mathrm{L}_\kappa^- = -\mathrm{K}_\kappa$, $\mathrm{L}_+^{\star\iota} = -\mathrm{K}^\iota$ such that $\mathrm{K}_\iota^\dagger = \mathrm{K}^\iota$, this can be written as

$$\mathrm{d}\hat{\varrho}(t) + \mathrm{K}_\kappa\hat{\varrho}(t)\,\mathrm{d}A_-^\kappa + \hat{\varrho}(t)\mathrm{K}^\iota\,\mathrm{d}A_\iota^+ = \left(\mathrm{L}_\kappa^j\hat{\varrho}(t)\mathrm{L}_j^{\star\iota} - \hat{\varrho}(t)\delta_\kappa^\iota\right)\mathrm{d}A_\iota^\kappa,$$

or in the notation above, $\mathrm{K}_+ = \mathrm{K}$, $\mathrm{K}^- = \mathrm{K}^\dagger$, $\mathrm{L}_+^k = \mathrm{L}^k$, $\mathrm{L}_k^{\star-} = \mathrm{L}_k$, $\mathrm{L}_k^{\star i} = \mathrm{L}_i^{k\dagger}$ as

$$\mathrm{d}\hat{\varrho}(t) + (\mathrm{K}\hat{\varrho}(t) + \hat{\varrho}(t)\mathrm{K}^\dagger - \mathrm{L}^j\hat{\varrho}(t)\mathrm{L}_j)\,\mathrm{d}t = \left(\mathrm{S}_\kappa^j\hat{\varrho}(t)\mathrm{S}_j^{\dagger i} - \hat{\varrho}(t)\delta_k^i\right)\mathrm{d}A_i^k$$
$$+\left(\mathrm{S}_k^j\hat{\varrho}(t)\mathrm{L}_j - K_k\hat{\varrho}(t)\right)\mathrm{d}A_-^k + \left(\mathrm{L}^j\hat{\varrho}(t)\mathrm{S}_j^{\dagger i} - \hat{\varrho}(t)\mathrm{K}^i\right)\mathrm{d}A_i^k,$$

with $\mathrm{K} + \mathrm{K}^\dagger = \mathrm{L}_j\mathbf{L}^j$, $\mathrm{L}^j = \mathrm{L}_j^\dagger$, $\mathbf{L}_i^{k\dagger}$ for any number of j's and arbitrary $\mathrm{K}^j = \mathrm{K}_j^\dagger$, L_k^i, $i, j, k = 1, \ldots, d$. This is the quantum stochastic generalization of the general form (Belavkin, 1988a) for the nonstochastic (Lindblad) Master equation corresponding to the case $d = 0$. In the case $d > 0$ with pseudounitary block-matrix $\mathrm{S} = [\mathrm{S}_\kappa^\iota]_{\nu=-,\circ,+}^{\iota=-,\circ,+}$ in the sense $\mathbf{S}^* = \mathbf{S}^{-1}$, it gives the general form of quantum stochastic Langevin equation corresponding to the HP unitary evolution for $\psi(t)$ (Hudson and Parthasarathy, 1984).

The nonlinear form of this decoherence equation for the exactly normalized density operator $\hat{\rho}(t) = \hat{\varrho}(t)/\mathrm{Tr}_{\mathfrak{h}}\hat{\varrho}(t)$ was obtained for different commutative Itô algebras in the literature (Barchielli and Belavkin, 1991; Belavkin, 1990c, 1992a).

REFERENCES

Barchielli, A. and Belavkin, V. P. (1991). *Measurements continuous in time and a posteriori states in quantum mechanics. Journal of Physics A: Mathematical and General* **24**, 1495–1514.

Belavkin, V. P. (1999). *Optimal Measurement and Control in Quantum Dynamical Systems*, Preprint No. 411, *Institute of Physics*, Nicholas Copernicus University, Torun', 1979, Feb. [Published in *Rep. Math. Phys.* **43**(3), 405–425,

Belavkin, V. P. (1980). Quantum filtering of Markov signals with Wight quantum noise. *Radiotechnika and Electronika* **25**, 1445–1453. [Full English translation in *Quantum Communications and Measurement*. Belavkin, V. P. *et al.*, eds., Plenum Press, New York, pp. 381–392, 1994.

Belavkin, V. P. (1985). In: *Information Complexity and Control in Quantum Physics*, A. Blaquière ed., Springer-Verlag, Udine, pp. 331–336.

Belavkin, V. P. (1988a). In: *Modelling and Control of Systems*, A. Blaquière, ed., Lecture Notes in Control and Information Sciences, Springer, Vol. 121, pp. 245–265.

Belavkin, V. P. (1988b). *A New Form and a ⋆-Algebraic Structure of Quantum Stochastic Integrals in Fock Space. Rediconti del Sem. Mat. e Fis. di Milano* **LVIII**, 177–193.

Belavkin, V. P. (1989). *A new wave equation for a continuous nondemolition measurement. Physics Letters A* **140**, 355–358.

Belavkin, V. P. (1990a). In: *Stochastic Methods in Experimental Sciences*, W. Kasprzak and A. Weron, eds., World Scientific, Singapore, pp. 26–42.

Belavkin, V. P. (1990b). *A Posterior Schrödinger Equation for Continuous Nondemolition Measurement*. *Journal of Mathematics and Physics* **31**, 2930–2934.

Belavkin, V. P. (1990c). Stochastic posterior equations for quantum nonlinear filtering. In *Prob. Theory and Math. Stat.*, B. Grigelionis *et al.* eds., VSP/Mokslas, Vilnius **1**, pp. 91–109.

Belavkin, V. P. (1991). *A quantum nonadapted itô formula and stochastic analysis in fock scale. J. Func. Anal.* **102**(2), 414–447.

Belavkin, V. P. (1992a). *Quantum stochastic calculus and quantum nonlinear filtering. Journal of Multivariate Analysis* **42**(2), 171–201.

Belavkin, V. P. (1992). *Quantum continual measurements and a posteriori collapse on CCR. Communications in Mathematical Physics* **146**, 611–635.

Belavkin, V. P. (1993). *The unified itô formula has the pseudo-poisson structure. Journal of Mathematics and Physics* **34**(4), 1508–1518.

Belavkin, V. P. (1994). *Nondemolition principle of quantum measurement theory. Foundations of Physics* **24**(5), 685–714.

Belavkin, V. P. (1995a). *A dynamical theory of quantum continuous measurements and spontaneous localizations. Russian Journal of Mathematical Physics* **3**(1), 3–24.

Belavkin, V. P. (1995b). *On stochastic generators of completely positive cocycles. Russian Journal of Mathematics and Physics* **3**, 523–528.

Belavkin, V. P. (1997). *Quantum stochastic positive evolutions: Charcterization, construction, dilation. Communications in Mathematical Physics* **184**, 533–566.

Belavkin, V. P. (1998). *On quantum itô algebras and their decompositions. Letters in Mathematical Physics* **45**, 131–145.

Belavkin, V. P. (2000a, April). In *New Development of Infinite-Dimensional Analysis and Quantum Probability*, RIMS, Kokyuroku, Vol. 1139, pp. 54–73.

Belavkin, V. P. (2000b). *Quantum stochastics, dirac boundary value problem, and the inductive stochastic limit. Rep. Math. Phys.* **46**(3), 359–382.

Belavkin, V. P. (2001). In *Evolution Equations and Their Applications in Physical and Life Sciences*, G. Lumer and L. Weis, eds., *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, Inc, New York, Vol. 215, pp. 311–327.

Belavkin, V. P. (2002). Quantum causality, stochastics, trajectories and information. *Reports on Progress in Physics* **65**, 353–420.

Belavkin, V. P. and Kolokoltsov, V. N. (2001). Stochastic evolutions as boundary value problems. In *Infinite Dimensional Analysis and Quantum Probability*, Research Institute for Mathematical Studies, Kokyuroku, Vol. 1227, pp. 83–95.

Belavkin, V. P. and Melsheimer, O. (1996). *A stochastic Hamiltonian approach for quantum Jumps, spontaneous localizations and continuous trajectories. Quantum and Semiclassical Optics* **8**, 167–187.

Belavkin, V. P. and Staszewski, P. (1989). *A quantum particle undergoing continuous observation. Physical Letters A* **140**, 359–362.

Belavkin, V. P. and Staszewski, P. (1992). *Nondemolition observation of a free Quantum particle. Physical Review A* **45**(3), 1347–1356.

Carmichael, H. J. (1993). *An Open System Approach to Quantum Optics*, Lecture Notes in Physics, m18, Springer, Berlin.

Carmichael, H. J. (1994). In *Quantum Optics VI*, J. D. Harvey and D. F. Walls, eds., Springer, Berlin.

Chruscinski, D. and Staszewski, P. (1992). *On the asymptotic solutions of the Belavkin's stochastic wave Equation. Physica Scripta.* **45**, 193–199.

Cramer, J. G. (1986). *Review of Modern Physics* **58**, 647–687.

Diosi, L. (1988). *Physical Review A* **40**, 1165–1174.

Dunn, R., Parkins, A. S., Zoller, P., and Gardiner, C. W. (1992). *Physical Review A* **46**, 4382.

Gardiner, C. W. (1991). *Quntum Noise*, Springer-Verlag, Berlin.

Gardiner, C. W. and Collett, M. J. (1985). *Physical Review A* **31**, 3761.
Gardiner, C. W., Parkins, A. S., and Zoller, P. (1992). *Physical Review A* **46**, 4363.
Gisin, N. (1984). *Quantum Measurement and Stochastic Proceses*. *Physical Review Letters* **52**(19), 1657–1660.
Gisin, N. (1989). *Helvetica Physica Acta* **62**, 363.
Gisin, N. and Percival, I. C. (1992). *The quantum state diffusion model applied to open systems. Journal of Physics A: Mathematical and General* **25**, 5677–5691.
Gisin, N. and Percival, I. C. (1993). *The quantum state diffusion picture of physical processes. Journal of Physics A: Mathematical and General* **26**, 2245–2260.
Goetsch, P. and Graham, R. (1993). *Quantum trajectories for nonlinear optical processes. Annalen der Physik* **2**, 708–719.
Goetsch, P. and Graham, R. (1994). *Linear stochastic wave equation for continuously measured quantum systems. Physical Review A* **50**, 5242–5255.
Goetsch, P., Graham, R., and Haake, F. (1995). *Schrödinger cat and single runs for the damped harmonic oscillator. Physical Review A* **51**(1), 136–142.
Hudson, R. L. and Parthasarathy, K. R. (1984). *Quantum itô's formula and stochastic evolution. Communications in Mathematical Physics* **93**, 301–323.
Itô, K. (1951). On a formula concerning stochastic differentials. *Nagoya Mathematical Journal* **3**, 55–65.
Kolokoltsov, V. N. (1995). *Scattering theory for the Belavkin equation describing a quantum particle with continuously observed coordinate. Journal of Mathematics and Physics* **36**(6), 2741–2760.
Lindblad, G. (1976). *On the generators of quantum stochastic semigroups. Communications of Mathematical Physics* **48**, 119–130.
Persival, I. (1999). *Quantum State Diffusion*, Cambridge University Press, Cambridge, UK.
Schrödinger, E. (1931). Sitzberg Press Akad. Wiss. Phys-Math. Kl. 144–153.
Stratonovich, R. L. (1966). *Conditional Markov Processes and Their Applications to Optimal Control*. Moscow State University, Moscow.
Wigner, E. (1963). *American Journal of Physics* **31**, 6.
Wiseman, H. M. and Milburn, G. J. (1993). *Physical Review A 47*, 642.
Wiseman, H. M. and Milburn, G. J. (1994). *Physical Review A* **49**, 1350.